福建中學

FUKIEN SECONDARY SCHOOL

S6 Mock Examination (2021-2022)

Mathematics Extended Part Module 2

(2 hours 30 minutes)

Date: 28th January 2022 Name:____________________

Time: 8:30a.m. - 11:00a.m. Class: ________ No.: ______

Instructions to students:

1. This paper consists of TWO sections, A and B.
2. Attempt ALL questions in Section A and Section B.
3. Write your answers in the spaces provided.
4. Unless otherwise specified, show your workings clearly.
5. Unless otherwise specified, numerical answers must be exact.
6. The diagrams in this paper are not necessarily drawn to scale.

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

Section A (50 marks)

1. Let $f(x) = \dfrac{1}{\sqrt{x+6}}$. Find $f'(x)$ from first principles. (4 marks)

2. Let n be a positive integer. In the expansion of $\left(x-\frac{3}{x}\right)^2(1+2x)^n$, the constant term is 750.

(a) Find n.

(b) Find the coefficient of x in the expansion of $\left(x-\frac{3}{x}\right)^2(1+2x)^n$.

(5 marks)

3. (a) Prove that $\sin 2x + \sin 4x + \sin 6x = 4\cos x \cos 2x \sin 3x$.

(b) Solve the equation $\sin 4\theta + \sin 8\theta + \sin 12\theta = 0$, where $0 \le \theta \le \frac{\pi}{4}$.

(6 marks)

4. (a) Using integration by parts, find $\int e^{2x} \cos \pi x \, dx$.

(b) Using integration by substitution, evaluate $\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{1-2x} \sin \pi x \, dx$.

(7 marks)

5. (a) Using mathematical induction, prove that $\displaystyle\sum_{k=n+1}^{2n} \frac{1}{(3k-2)(3k+1)} = \frac{n}{(3n+1)(6n+1)}$ for all positive integers n.

(b) Using (a), evaluate $\displaystyle\sum_{k=41}^{90} \frac{1}{(3k-2)(3k+1)} - \sum_{k=81}^{90} \frac{1}{(3k-2)(3k+1)}$.

(7 marks)

6. Let $f(x)$ be a continuous function defined on $\mathbf{R}^+$, where $\mathbf{R}^+$ is the set of positive real numbers. Denote the curve $y = f(x)$ by H. It is given that H passes through the point $P(4, 7)$ and $f'(x) = \dfrac{x^2 - 3x + 4}{2x}$ for all $x > 0$.

(a) Does H have any maximum or minimum points? Explain your answer.

(b) Find the equation of H.

(c) Find the point(s) of inflexion of H.

(7 marks)

7. Consider the system of linear equations in real variables x, y, z

$$(E):\begin{cases} x-y-z=\beta \\ 2x+\alpha y+\alpha z=2\beta \\ 5x+(\alpha-2)y+(2\alpha+3)z=7\beta \end{cases}\quad \text{, where } \alpha,\ \beta\in\mathbf{R}.$$

(a) Assume that (E) has a unique solution.

(i) Find the range of values of α.

(ii) Express y in terms of α and β.

(b) Assume that $\alpha=-5$. If (E) is inconsistent, find the range of values of β.

(7 marks)

8. Denote the graph of $y = e^{\frac{x}{2}}$ and the graph of $y = -e^{-\frac{x}{2}}$ by F and G respectively, where $x > 0$. Let P be a moving point on F. The vertical line passing through P cuts G at the point Q. Denote the x-coordinate of P by u. The horizontal line passing through P cuts the y-axis at the point R while the horizontal line passing through Q cuts the y-axis at the point S. It is given that the area of the rectangle $PQSR$ increases at a constant rate of e^2 square units per minute. Find the rate of change of

(a) the length of PR when $u = 2$,

(b) the perimeter of the rectangle $PQSR$ when $u = 2$.

(7 marks)

Section B (50 marks)

9. Consider the curve G: $y=\sqrt{\dfrac{8-x^2}{3}}$, where $0<x<2\sqrt{2}$. Denote the tangent to G at $x=2$ by L.

   (a) Find the equation of L. (3 marks)

   (b) Let C be the curve $y=\sqrt{4-x^2}$, where $0 < x < 2$. It is given that L is a tangent to C. Find

   (i) the point(s) of contact of L and C;

   (ii) the point(s) of intersection of C and G;

   (iii) the area of the region bounded by L, C and G.

   (9 marks)

10. (a) Let $f(x)$ be a continuous function defined on the interval $[0, a]$, where a is a positive constant. Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$. (3 marks)

(b) Prove that $\int_0^{\frac{\pi}{3}} \ln(3\tan x + \sqrt{3})dx = \int_0^{\frac{\pi}{3}} \ln\left(\frac{12}{3\tan x + \sqrt{3}}\right)dx$. (3 marks)

(c) Using (b), prove that $\int_0^{\frac{\pi}{3}} \ln(3\tan x + \sqrt{3})dx = \frac{\pi \ln 12}{6}$. (3 marks)

(d) Evaluate $\int_0^{\frac{\pi}{3}} \frac{x\sec x}{\sqrt{3}\sin x + \cos x}dx$. (4 marks)

11. Let m, n and k be real numbers such that $m - n + 2k \neq 0$. Define $A = \begin{pmatrix} m-k & m-n+k \\ k & n-k \end{pmatrix}$,

$X = \dfrac{1}{-m+n-2k}(A - mI)$ and $Y = \dfrac{1}{m-n+2k}[A-(n-2k)I]$, where I is the 2×2 identity matrix.

(a) Evaluate XY, YX and $X + Y$. (3 marks)

(b) Prove that $X^2 = X$ and $Y^2 = Y$. (2 marks)

(c) Prove that $A^p = (n-2k)^p X + m^p Y$ for all positive integers p. (3 marks)

(d) Evaluate $\begin{pmatrix} 6 & 2 \\ 0 & 4 \end{pmatrix}^{2016}$. (4 marks)

12. The position vectors of the points A, B, C and D are $6\mathbf{i} + 8\mathbf{j} + m\mathbf{k}$, $5\mathbf{i} + (m + 1)\mathbf{j} + 21\mathbf{k}$, $6\mathbf{i} + m\mathbf{j} + 17\mathbf{k}$ and $11\mathbf{i} + (m + 2)\mathbf{j} - \mathbf{k}$ respectively, where $m \in \mathbf{R}$. Suppose that $\overrightarrow{AB}$ is perpendicular to $2\mathbf{i} + 7\mathbf{j} - \mathbf{k}$. Denote the plane which contains A, B and C by P, and a unit vector which is perpendicular to P by $\mathbf{v}$.

(a) Find

(i) m,

(ii) $\mathbf{v}$.

(5 marks)

(b) Let E be the projection of D on P.

(i) Find $\overrightarrow{ED}$.

(ii) Find the shortest distance from D to P.

(iii) Find the position vector of E.

(iv) Is E the orthocentre of $\triangle ABC$? Explain your answer.

(8 marks)

END OF PAPER