FUKIEN SECONDARY SCHOOL

S5 Final Examination (2020-2021)

Mathematics Extended Part Module 2

(2 hours 30 minutes)

Date: 22nd June 2021 Name:____________________

Time: 8:30 a.m.-11:00 a.m. Class: ________ No.: ______

Instructions to students:

1. This paper consists of TWO sections, A and B.
2. Attempt ALL questions in Section A and Section B.
3. Write your answers in the Answer Book provided.
4. Unless otherwise specified, show all workings clearly.
5. Unless otherwise specified, numerical answers must be exact.
6. The diagrams in this paper are not necessarily drawn to scale.

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A\cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A\cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A\sin B = \cos(A-B) - \cos(A+B)$	

Section A (50 marks)

1. Find $\dfrac{d}{dx}\tan 5x$ from first principles. (5 marks)

2. Let n be a positive integer and a be a non-zero constant. In the expansion of $(1 + ax)^n$, the ratio of the coefficient of x^5 to that of x^3 is $3a^2 : 5$.

(a) Find the value of n.

(b) Express the constant term in the expansion of $(1+ax)^n\left(2+\frac{1}{x}\right)^2$ in terms of a.

(5 marks)

3. (a) If $(\sqrt{3}-\sqrt{2})\tan A=(\sqrt{3}+\sqrt{2})\tan B$, prove that $\sqrt{2}\sin(A+B)=\sqrt{3}\sin(A-B)$.

(b) Using (a), solve the equation $(\sqrt{3}-\sqrt{2})\tan\left(x+\frac{2\pi}{5}\right)=(\sqrt{3}+\sqrt{2})\tan\left(x+\frac{3\pi}{20}\right)$,

where $0\le x\le\frac{\pi}{2}$.

(5 marks)

4. (a) Using integration by parts, find $\int x^k \ln x \, dx$, where k is a positive constant.

(b) Consider the curve Γ: $y = \sqrt{x} \ln x$, where $1 \le x \le 4$. Let R be the region bounded by Γ, the x-axis, and the lines $x = 1$ and $x = 4$.

(i) Find the area of R.

(ii) Find the volume of the solid of revolution generated by revolving R about the x-axis.

(7 marks)

5. (a) Using mathematical induction, prove that

$$\sum_{k=1}^{n} \frac{1}{(3k-2)(3k+1)} = \frac{n}{3n+1}$$

for all positive integers n.

(b) Using (a), evaluate $\displaystyle\sum_{k=2}^{97} \frac{3}{(3k-2)(3k+1)}$.

(6 marks)

6. Consider the curve Γ: $y = \frac{1}{2}\sqrt{20 - x^2}$, where $0 < x < 2\sqrt{5}$. Denote the tangent to Γ at $x = 4$ by L.

(a) Find the equation of L.

(b) Let C be the curve $y = \sqrt{\frac{25}{2} - x^2}$, where $0 < x < \frac{5\sqrt{2}}{2}$. It is given that L is a tangent to C.

(i) Find the point(s) of contact of L and C.

(ii) $P(u , v)$ is a moving point on C such that u increases at a constant rate of 2 units per second. Find the rate(s) of change of v when P is at the point(s) of contact of L and C.

(7 marks)

7. Consider the following system of linear equations in real variables x, y, z

$$(E):\ \begin{cases} x + 2y + hz = k \\ 2x - 3y + z = 12 \\ 4x + y - z = 2 \end{cases}, \text{ where } h, k \in \mathbb{R}.$$

(a) Assume that (E) has a unique solution.

(i) Find the range of values of h.

(ii) Express z in terms of h and k.

(b) Assume that (E) has infinitely many solutions. Solve (E).

(6 marks)

8. ABC is a triangle. D is the mid-point of AC. E is a point lying on BC such that $BE : EC = 1 : r$. AB is produced to the point F such that DEF is a straight line. It is given that $DE : EF = 1 : 4$. Let $\overrightarrow{OA} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}$, $\overrightarrow{OB} = -\mathbf{i} + 5\mathbf{j} - \mathbf{k}$ and $\overrightarrow{OC} = 3\mathbf{i} - 3\mathbf{j}$, where O is the origin.

(a) By expressing $\overrightarrow{AE}$ and $\overrightarrow{AF}$ in terms of r, find r.

(b) (i) Find $\overrightarrow{AD} \cdot \overrightarrow{DE}$.

(ii) Are B, D, C and F concyclic? Explain your answer.

(9 marks)

Section B (50 marks)

9. Let $f(x) = \dfrac{(x-2)^2 - 7}{x+2}$ for all real numbers $x \neq -2$. Denote the graph of $y = f(x)$ by G.

   (a) Find the asymptote(s) of G. (3 marks)

   (b) Find $f'(x)$. (1 mark)

   (c) Find the maximum point and the minimum point of G. (4 marks)

   (d) Let R be the region bounded by G, the x-axis, the y-axis and the vertical line passing through the minimum point of G. Find the volume of the solid of revolution generated by revolving R about the x-axis. (4 marks)

10. (a) Prove that $\dfrac{1+\sin 2t}{1+\cos 2t} = \dfrac{1}{2}\sec^2 t + \tan t$. (2 marks)

(b) Let $f(x)$ be a differentiable function and $f'(x)$ be the first derivative of $f(x)$ with respect to x. Prove that $\int e^x [f(x) + f'(x)]\,dx = e^x f(x) + C$, where C is any constant. (2 marks)

(c) Using (a) and (b), evaluate $\int_0^{\frac{\pi}{2}} \dfrac{e^x(1+\sin x)}{1+\cos x}\,dx$. (4 marks)

(d) Evaluate $\int_0^{\frac{\pi}{2}} \dfrac{1+\cos x}{e^x(1+\sin x)}\,dx$. (4 marks)

11. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ a(1-c) & c-b & 1 \\ ab(1-c) & -b^2 & b+c \end{pmatrix}$ and $M = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ ab & b & 1 \end{pmatrix}$, where a, b and c are real numbers.

(a) (i) Express M^{-1} in terms of a and b.

(ii) Express $M^{-1}AM$ in terms of c.

(iii) Let λ be a non-zero real number. Prove that $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}$ for all positive integers n.

(10 marks)

(b) Using (a), evaluate $\begin{pmatrix} 2 & 0 & 0 \\ 4 & 2 & 2 \\ 8 & -8 & 10 \end{pmatrix}^{99}$.

(3 marks)

12. Let $\overrightarrow{OA} = \mathbf{i} - 2\mathbf{j}$, $\overrightarrow{OB} = -7\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and $\overrightarrow{OC} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, where O is the origin.

(a) (i) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

(ii) Hence, find the volume of the tetrahedron $OABC$.

(5 marks)

(b) Denote the plane containing A, B and C by Π. Let $\overrightarrow{OP} = -4\mathbf{j} + 8\mathbf{k}$ and $\overrightarrow{OQ} = -9\mathbf{i} + 2\mathbf{j} - 10\mathbf{k}$. The straight line passing through P and Q intersects Π at D.

(i) Is $\overrightarrow{PQ}$ perpendicular to Π? Explain your answer.

(ii) Find $PD : DQ$.

(iii) Describe the geometric relationship between AB, CD and PQ. Explain your answer.

(8 marks)

— End of Paper —