FUKIEN SECONDARY SCHOOL

S4 Final Examination (2020-2021)

Mathematics Extended Part Module 2

(2 hours)

Date: 22nd June 2021 Name:____________________

Time: 8:30 a.m.-10:30 a.m. Class: ________ No.: ______

Instructions to students:

1. This paper consists of TWO sections, A and B.
2. Attempt ALL questions in Section A and Section B.
3. Write your answers in the Answer Book provided.
4. Unless otherwise specified, show all workings clearly.
5. Unless otherwise specified, numerical answers must be exact.

FORMULAS FOR REFERENCE

$\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan (A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2 \sin A \cos B = \sin (A + B) + \sin (A - B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2 \cos A \cos B = \cos (A + B) + \cos (A - B)$	
$2 \sin A \sin B = \cos (A - B) - \cos (A + B)$	

Section A (58 marks)

1. Let $f(x)=\dfrac{7x}{4+3x^2}$. Prove that $f(1+h)-f(1)=\dfrac{h-3h^2}{7+6h+3h^2}$. Hence find $f'(1)$ from first principles.

(4 marks)

2. In the expansion of $(1-4x)^n$, the sum of the coefficients of x and x^2 is 540, where n is a positive integer.
   (a) Find the value of n.
   (b) Find the coefficient of x^3.

(4 marks)

3. Let $y = e^{3x} \sin x$.

   (a) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

   (b) Let k be a constant. If $2\dfrac{d^2y}{dx^2} - 3k^2\dfrac{dy}{dx} + 20y = 0$ for all real values of x, find the values of k.

(5 marks)

4. Let $h(x)$ be a continuous function defined on $\mathbf{R}^+$, where $\mathbf{R}^+$ is the set of positive real numbers. It is given that $h'(x) = \dfrac{1}{x}(x^2 - 2x + 25)$ for all $x > 0$.

   (a) Is $h(x)$ an increasing function? Explain your answer.

   (b) Denote the graph of $y = h(x)$ by H. Prove that H has only one point of inflexion.

(5 marks)

5. (a) Evaluate $\lim_{x \to 0} \frac{e^{5x} \sin 2x}{e^{3x} - e^{-3x}}$.

(b) Evaluate $\lim_{x \to \infty} x \ln\left(1 + \frac{2}{x} - \frac{3}{x^2}\right)$.

(6 marks)

6. (a) Prove that $\sin(A - B)\sin(A + B) = \cos^2 B - \cos^2 A$.

(b) Hence solve the equation $\sin\left(x - \frac{\pi}{6}\right)\sin\left(x + \frac{\pi}{6}\right) + \sin\left(x - \frac{\pi}{3}\right)\sin\left(x + \frac{\pi}{3}\right) = 0$,

where $0 \le x < \pi$.

(6 marks)

7. Define $f(x) = x\ln x$ for all $x \in (0,\ 100)$. Denote the graph of $y = f(x)$ by F.

   (a) Prove that F has only one minimum point.

   (b) Let O and M be the origin and the minimum point of F respectively. If F cuts the x-axis at N, find the area of $\triangle OMN$.

(6 marks)

8. (a) Using mathematical induction, prove that $\sum_{r=1}^{n}(r+2)(1-r) = -\frac{n(n-1)(n+4)}{3}$ for all positive integers n.

   (b) Using (a), evaluate $\sum_{r=445}^{666}\left(\frac{r+2}{1\ 036}\right)\left(\frac{r-1}{301}\right)$.

(7 marks)

9. The following table shows the characteristics of a polynomial function $f(x)$ on different intervals of x.

x	$x<-5$	$x=-5$	$-5<x<-2$	$x=-2$	$-2<x<1$	$x=1$	$x>1$
$f(x)$	/	-54	/	0	/	54	/
$f'(x)$	$-$	0	$+$	$+$	$+$	0	$-$
$f''(x)$	$+$	$+$	$+$	0	$-$	$-$	$-$

(a) Find the range of values of x for which the curve $y=f(x)$ is increasing.

(b) Find the range of values of x for which the curve is concave downwards.

(c) Find the maximum / minimum point(s) and point(s) of inflexion of the curve.

(d) Sketch the curve $y=f(x)$.

(7 marks)

10. (a) Using mathematical induction, prove that $\sin x\sum_{k=1}^{n}\sin 2kx = \sin nx \sin (n+1)x$ for all positive integers n.

(b) Using the result of (a), evaluate $\sum_{k=1}^{369}\sin^2\left(\frac{k\pi}{5}-\frac{\pi}{4}\right)$.

(8 marks)

Section B (42 marks)

11. A function $f(x) = ax^3 + bx^2 + x$ is given, where a and b are constants. Figure 1 shows a sketch of the graph of $y = f'(x)$.

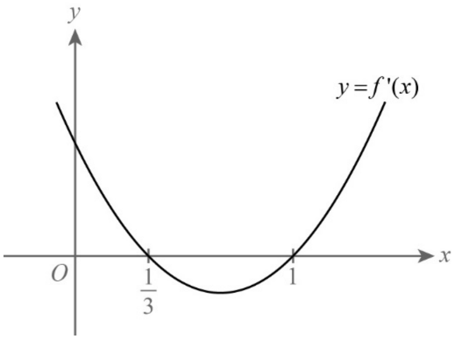

Figure 1

(a) Find the values of a and b.

(3 marks)

(b) Find the x- and y-intercepts of the graph of $y = f(x)$.

(2 marks)

(c) Find the maximum/minimum point(s) and point(s) of inflexion of the graph of $y = f(x)$.

(5 marks)

(d) Sketch the graph of $y = f(x)$.

(2 marks)

12. In Figure 2, PQ is a straight coastline of 12 km long. S and F are islands which are 3 km away from P and Q respectively, where $PS \perp PQ$ and $QF \perp PQ$. M and N are points on PQ such that $PM = x$ km and $NQ = 4$ km.

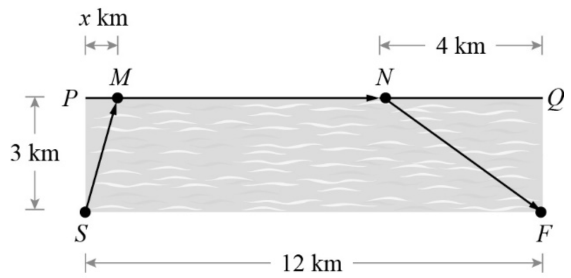

Figure 2

In a cross country competition, a team sails at a constant speed of 5 km/h along a straight path from S to M, and then cycles at a constant speed of 12 km/h along the coastline to N. Finally, the team sails at a constant speed of $\dfrac{300}{20-x}$ km/h, from N to the finish point F. Let T hours be the time taken by the team to complete the competition.

(a) (i) Express T in terms of x.

(ii) Find the minimum value of T and the corresponding distance of SM.

(6 marks)

(b) In Figure 3, a camera crew stays at C for filming the team, where C lies on SF and the length of SC is $\sqrt{3}$ times the length of SM obtained in (a)(ii). Let $\angle RCN = \alpha$ and $\angle RNC = \beta$, where R is the position of the team on their way when they are cycling.

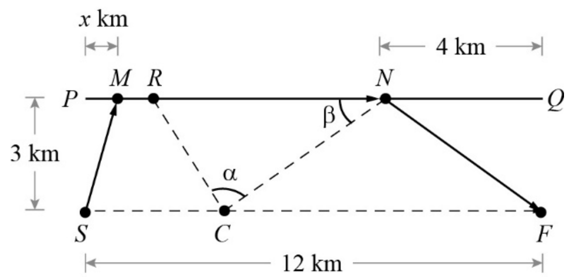

Figure 3

(i) By finding $\sin\beta$ and $\cos\beta$, show that $RN = \dfrac{13\tan\alpha}{2\tan\alpha + 3}$ km.

(ii) Find the rate of change of α (in radian/s) when $\alpha = 1$ radian.
(Give your answer correct to 3 significant figures.)

(7 marks)

----

13. (a) (i) If $7\sin\theta - 24\cos\theta$ is expressed in the form $r\sin(\theta - A)$ where $r > 0$ and $0° < A < 90°$, find r and A.

(ii) Let $y = 14\sin\theta - 48\cos\theta + 14$.
Using the result in (i), find the maximum and minimum values of y.

Find also the value(s) of θ for $0° < \theta < 360°$ at which y attains its maximum.

(10 marks)

(b) α and β are two acute angles satisfying the equation
$6\cos^2\theta - 5\cos\theta + 1 = 0$.
Without solving the equation, show that

$$\cos\frac{\alpha+\beta}{2} + \cos\frac{\alpha-\beta}{2} = \sqrt{2}.$$

[Hint: $\cos^2\frac{x}{2} = \frac{1}{2}(1+\cos x)$.]

(7 marks)

— End of Paper —