福建中學

# FUKIEN SECONDARY SCHOOL

S6 Mock Examination (2020-2021)

Mathematics Extended Part Module 2

(2 hours 30 minutes)

Date: 25th January 2021

Time: 8:30a.m. - 11:00a.m.

Name:____________________

Class: ________ No.: ______

**Instructions to students**:

1. This paper consists of TWO sections, A and B.
2. Attempt ALL questions in Section A and Section B.
3. Write your answers in the spaces provided.
4. Unless otherwise specified, show your workings clearly.
5. Unless otherwise specified, numerical answers must be exact.
6. The diagrams in this paper are not necessarily drawn to scale.

**FORMULAS FOR REFERENCE**

| | |
|---|---|
| $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ | $\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$ |
| $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$ | $\sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$ |
| $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$ | $\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$ |
| $2\sin A \cos B = \sin(A+B) + \sin(A-B)$ | $\cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$ |
| $2\cos A \cos B = \cos(A+B) + \cos(A-B)$ | |
| $2\sin A \sin B = \cos(A-B) - \cos(A+B)$ | |

## Section A (50 marks)

1. Let $f(x) = \dfrac{x^3}{4-5x^2}$. Find $f'(2)$ from first principles.

(4 marks)

2. Let $P(x)=\begin{vmatrix} 3 & 2 & (x-\lambda)^2 \\ 4 & x-\lambda & 0 \\ (x-\lambda)^3 & 5 & 1 \end{vmatrix}$, where $\lambda$ is a positive real number. It is given that the coefficient of $x^2$ in the expansion of $P(x)$ is –1195. Find

(a) $\lambda$,

(b) $P''(\lambda)$.

(6 marks)

3. Let $y = x^2 \ln x$, where $x > 0$.

(a) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

(b) Let $k$ be a constant such that $x^2 \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + kx\dfrac{\mathrm{d}y}{\mathrm{d}x} + 4y = 0$ for all positive values of $x$.

Find the value of $k$.

(5 marks)

4. (a) If $\cot A = 3\tan B$, prove that $2\cos(A+B) = \cos(A-B)$.

(b) Using (a), solve the equation $\cot\left(x+\frac{11\pi}{15}\right) = 3\tan\left(x-\frac{4\pi}{15}\right)$, where $0 \le x \le \frac{\pi}{2}$.

(6 marks)

5. (a) Using mathematical induction, prove that $\sum_{k=1}^{n}(2k-1)^3 = n^2(2n^2-1)$ for all positive integers $n$.

(b) Using (a), evaluate $\sum_{k=3}^{50}(2k-1)^3$ .

(7 marks)

6. (a) Using integration by substitution, find $\int x^5\sqrt{x^3+9}dx$, where $x \geq -\sqrt[3]{9}$.

(b) At any point $(x, y)$ on the curve $\Gamma$, the slope of the tangent to $\Gamma$ is $10x^5\sqrt{x^3+9}$. The $y$-intercept of $\Gamma$ is $-100$. Find the equation of $\Gamma$.

(7 marks)

7. In Figure 1, a car, $X$, is travelling from $O$ to $A$ at a constant speed of 16 m $s^{-1}$, while another car, $Y$, is travelling from $B$ to $O$ at a constant speed of 18 m $s^{-1}$. $\angle AOB = \frac{\pi}{3}$. Let $x$ m be the distance between $X$ and $O$, $y$ m be the distance between $Y$ and $O$, and $k$ m be the distance between $X$ and $Y$.

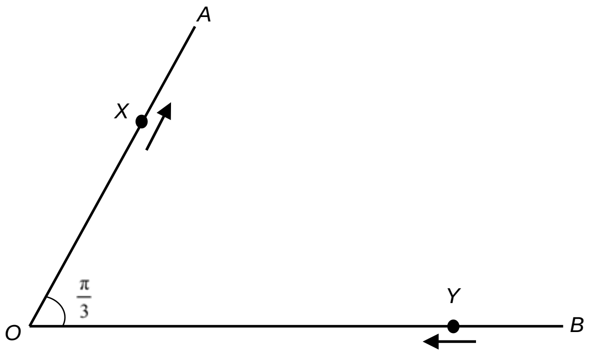


Figure 1

(a) Express $k^2$ in terms of $x$ and $y$.

(b) Find the rate of change of the distance between $X$ and $Y$ when $x = 300$ and $y = 800$.

(7 marks)

8. Consider the system of linear equations in real variables $x$, $y$, $z$

$$(E):\begin{cases} x - y + z = 2k \\ 3x + 5y + (h+1)z = 4 \\ 4x + hy + (2h-3)z = k+2 \end{cases}$$

, where $h$ and $k$ are real numbers.

(a) Assume that $(E)$ has a unique solution.

(i) Prove that $h \neq 6$ and $h \neq 8$.

(ii) Express $z$ in terms of $h$ and $k$.

(b) Assume that $h = 8$ and $(E)$ is consistent.

(i) Find $k$.

(ii) Solve $(E)$.

(8 marks)

## Section B (50 marks)

9. Let $f(x) = \dfrac{x^2 + cx - 5}{x+3}$, where $c$ is a constant and $x \neq -3$. Denote the graph of $y = f(x)$ by $G$. It is given that the oblique asymptote of $G$ passes through the point (8 , 1).

(a) Find the value of $c$. (2 marks)

(b) Find $f'(x)$. (2 marks)

(c) Find the maximum point and the minimum point of $G$. (4 marks)

(d) Let $R$ be the region bounded by $G$, the line $y = x + 9$, the line $x = m$ and the $y$-axis, where $m > 0$. Is the area of $R$ less than $16m$? Explain your answer. (4 marks)

10. (a) (i) Prove that $\int \tan^4 x dx = \frac{1}{3}\tan^3 x - \int \tan^2 x dx$.

(ii) Evaluate $\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \tan^4 x dx$.

(5 marks)

(b) (i) Let $f(x)$ be a continuous function for $a \le x \le b$, where $a$ and $b$ are constants, such that $f(x) = f(a+b-x)$ for all $a \le x \le b$. Prove that $\int_a^b xf(x)dx = \frac{a+b}{2}\int_a^b f(x)dx$.

(ii) Evaluate $\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} x\tan^4 x dx$.

(5 marks)

(c) Consider the curve $\Gamma$: $y = \sqrt{x}\tan^2 x$, where $\frac{7\pi}{4} \le x \le \frac{9\pi}{4}$. Let $R$ be the region bounded by $\Gamma$, the $x$-axis, the two lines $x = \frac{7\pi}{4}$ and $x = \frac{9\pi}{4}$. Find the volume of the solid of revolution generated by revolving $R$ about the $x$-axis.

(3 marks)

11. Let $A=\begin{pmatrix}9 & 4\\ 4 & 3\end{pmatrix}$ and $P=\begin{pmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}$.

(a) (i) Evaluate $P^tP$.

(ii) Evaluate $P^{-1}AP$.

(5 marks)

(b) For any matrix $\begin{pmatrix}x\\ y\end{pmatrix}$, define $\left\|\begin{pmatrix}x\\ y\end{pmatrix}\right\|=\sqrt{x^2+y^2}$.

(i) For any matrix $\begin{pmatrix}x\\ y\end{pmatrix}$, prove that $\left\|P\begin{pmatrix}x\\ y\end{pmatrix}\right\|=\left\|\begin{pmatrix}x\\ y\end{pmatrix}\right\|$.

(ii) By considering $\begin{pmatrix}u\\ v\end{pmatrix}=P^{-1}\begin{pmatrix}x\\ y\end{pmatrix}$, prove that $\left\|A\begin{pmatrix}x\\ y\end{pmatrix}\right\|\le 11\left\|\begin{pmatrix}x\\ y\end{pmatrix}\right\|$ for any matrix $\begin{pmatrix}x\\ y\end{pmatrix}$.

(7 marks)

12. Let $\overrightarrow{OP} = 5\mathbf{i} + t\mathbf{j} + \mathbf{k}$ , $\overrightarrow{OQ} = \mathbf{i} + \mathbf{j} - t\mathbf{k}$ and $\overrightarrow{OA} = 2\mathbf{j} + 2\mathbf{k}$ , where $t$ is a constant and $O$ is the origin. It is given that $A$ is equidistant from $P$ and $Q$.

(a) Find $t$.

(3 marks)

(b) Let $\overrightarrow{OR} = -\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\overrightarrow{OS} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ . Denote the plane which contains $P$, $Q$ and $R$ by $\Pi$.

(i) Find a unit vector which is perpendicular to $\Pi$.

(ii) Find the angle between $RS$ and $\Pi$.

(iii) It is given that $T$ is a point lying on $\Pi$ such that $\overrightarrow{ST}$ is perpendicular to $\Pi$. Let $U$ be a point such that $\overrightarrow{AU} = 2\overrightarrow{AQ} - 2\overrightarrow{AR}$ . Describe the geometric relationship between $S$, $T$ and $U$. Explain your answer.

(10 marks)

End of Test